Transformation of a system of higher order linear differential equations to a system of linear differential equations of the first order with applications

By Eman Talib .M. AL-katheri

King Abdul Aziz University
17-December-2013

Abstract

The objective, of the present work, is to give a brief account on the transformation of a system of a higher order ordinary differential equations to another system of first order linear differential equations with certain applications. We are interested in the case of constant coefficients. We started by presenting some guide informations about the classification of the first order linear systems of equations. Then we explained how to transform a single second order linear differential equation to a system of two linear differential equations of the first order and we gave some applications on the different cases.

We considered the transformation of a system of two second order linear differential equations to a system of four linear differential equations of the first order. This is done through the physical application of the coupled pendula. We determined the matrix of coefficients and calculated its eigen values and the corresponding eigen vectors. The solution of the original problem is obtained and drawn.

(Dedication)

To my father and my mother.

(Acknowledgment)

My thanks go to Dr. Wafaa Al-Barakati of the faculty of Science King Ablul Aziz University, who helped me in the completion of this research, with best wishes for continued health and wellness

Index

Title Page

Introduction 6

Transformation of a second order linear differential equation to a system of two linear differential equations of the first order 8

Solution of a Homogeneous System of Two Linear First Order Equations in Two Unknowns 10

Transformation of a system of second order linear differential equations to a system of four linear differential equations of the first order 14

The coupled pendula 15

Conclusion 22

Reference 23

Introduction

We will now turn our attention to solving systems of simultaneous homogeneous first order linear differential equations. we have to limit ourselves to the simplest instances: those systems of two equations and two unknowns only. But first, we shall have a brief overview and learn some notations and terminology.
A system of n linear first order differential equations in n unknowns (an $n \times n$ system of linear equations) has the general form:

$$\begin{array}{lcl} x_1' & = & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + g_1 \\ x_2' & = & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + g_2 \\ x_3' & = & a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n + g_3 \\ \vdots & & \vdots \qquad\qquad\qquad \vdots \\ \vdots & & \vdots \qquad\qquad\qquad \vdots \\ x_n' & = & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + g_n \end{array} \qquad (*)$$

Where the coefficients a_{ij}'s, and g_i's are constants. If every Term g_i is constant zero, then the system is said to be homogeneous. Otherwise, it is a nonhomogeneous system if even one of the g's is nonzero.
The system (*) is most often given in a shorthand format as a matrix-vector equation, in the form:

$$x' = Ax + g$$

$$\underset{x'}{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix}} = \underset{A}{\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}} + \underset{g}{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ g_n \end{bmatrix}}$$

Where the matrix of coefficients, $\boldsymbol{A}$, is called the *coefficient matrix* of the system. The vectors $\boldsymbol{x}'$, $\boldsymbol{x}$, and $\boldsymbol{g}$ are

$$x' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ g_n \end{bmatrix}.$$

For a homogeneous system, $\boldsymbol{g}$ is the zero vector. Hence it has the form:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

Transformation of a second order linear differential equation to a system of two linear differential equations of the first order

We give some simple examples to explain the process

Examples:

(i)The mechanical vibration equation $m\,u'' + \gamma\,u' + k\,u = F(t)$ is equivalent to the following system

$$x_1' = x_2$$

$$x_2' = \frac{-k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F(t)}{m}$$

Note that the system would be homogeneous (respectively, nonhomogeneous) if the original equation is homogeneous (respectively, nonhomogeneous).

(ii)The equation $y''' - 2y'' + 3y' - 4y = 0$ is equivalent to following system

$x_1' = x_2$
$x_2' = x_3$
$x_3' = 4\,x_1 - 3\,x_2 + 2\,x_3$

This process can be easily generalized. Given an n-th order linear equation of the form

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \ldots + a_2 y'' + a_1 y' + a_0 y = g(t)$.

Make the substitutions: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, … , $x_n = y^{(n-1)}$, and $x_n' = y^{(n)}$.

The first $n - 1$ equations follow thusly. Lastly, substitute the x's into the original equation to rewrite it into the n-th equation and obtain the system of the form:

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$\vdots \quad \vdots \quad \vdots$$
$$\vdots \quad \vdots \quad \vdots$$
$$x_{n-1}' = x_n$$
$$x_n^1 = \frac{-a_0}{a_n} x_1 - \frac{a_1}{a_n} x_2 - \frac{a_2}{a_n} x_3 - \cdots - \frac{a_{n-1}}{a_n} x_n + \frac{g(t)}{a_n}$$

Exercises:

1 – 3 Convert each linear equation into a system of first order equations.

1. $y'' - 4y' + 5y = 0$

Answers:

1.$x_1' = x_2$
$x_2' = -5x_1 + 4x_2$

2.$y''' - 5y'' + 9y = t \cos 2t$

Answers

2.$x_1' = x_2$
$x_2' = x_3$
$x_3' = -9x_1 + 5x_3 + t \cos 2t$

3.$y_{(4)} + 3y''' - \pi y'' + 2\pi y' - 6y = 11$

Answers

$x_1' = x_2$3.
$x_2' = x_3$
$x_3' = x_4$
$x_4' = 6x_1 - 2\pi x_2 + \pi x_3 - 3x_4 + 11$

Solution of a Homogeneous System of Two Linear First Order Equations in Two Unknowns

Given:

$$\boldsymbol{\mathcal{X}}' = \mathbf{A}\boldsymbol{\mathcal{X}}.$$

We first find the two eigenvalues, *r*, and their respective corresponding
eigenvectors, ***k***, of the coefficient matrix ***A***. Depending on the eigen values and eigenvectors, the general solution is written. Three different cases arise

Case I. Two distinct real eigenvalues r_1 and *r*2 of the matrix A:
The solution is

$$\mathcal{X} = c_1 k_1 e^{r_1 t} + c_2 k_2 e^{r_2 t}$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}.$$

This coefficient matrix has the eigenvalues *r*= −1 and 6. And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

Therefore, a general solution of this system of differential equations is:

$$\mathcal{X} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

Case II. Two complex conjugate eigenvalues $\lambda \pm \mu i$, where $\lambda + \mu i$ has as an eigen vector $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$:
The solution is

$$\mathcal{X} = c_1 e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) + c_2 e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}.$$

The characteristic equation is $r^2 + 1 = 0$, giving eigenvalues $r = \pm i$.
That is, $\lambda = 0$ and $\mu = 1$.
Take the first (the one with positive imaginary part) eigenvalue $r = i$, and find one of its eigenvectors:

$$(\boldsymbol{A} - r\,\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Solving the first equation of the system: $(2 - i)\, x_1 - 5x_2 = 0$, we get the relation $(2 - i)\, x_1 = 5x_2$. Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i = a + bi$$

Therefore, the general solution is

$$\mathcal{X} = c_1 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin(t)\right) + c_2 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos(t)\right)$$

$$= c_1\left(\begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + c_2\left(\begin{bmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}\right)\right.$$

Case III. A repeated real eigenvalue r:
There are two cases

(i) When two linearly independent eigenvectors exist –

$$\mathcal{X} = c_1 k_1 e^{rt} + c_2 k_2 e^{rt}$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalue is $r = 2$ (repeated). There are 2 sets of linearly independent eigenvectors, which could be represented by any 2 nonzero vectors that are not constant multiples of each other. For example:

$$k_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Therefore, a general solution is

$$\mathcal{X} = \mathrm{c}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{2\mathrm{t}} + \mathrm{c}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{2\mathrm{t}}$$

(ii) When only one linearly independent eigenvector exist –

$$\mathcal{X} = \mathrm{c}_1 \mathrm{k}_1 \mathrm{e}^{\mathrm{rt}} + \mathrm{c}_2(\mathrm{k}t\mathrm{e}^{\mathrm{rt}} + \eta \mathrm{e}^{\mathrm{rt}})$$

Note: Solve the system $(\boldsymbol{A} - r\,\boldsymbol{I})\,\boldsymbol{\eta} = \boldsymbol{k}$ to find the vector $\boldsymbol{\eta}$.

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \boldsymbol{x} \qquad , \qquad \boldsymbol{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenvalue is $r = -3$ (repeated). The corresponding system is

$$(\boldsymbol{A} - r\,\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} x = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Both equations of the system are $4x_1 - 4x_2 = 0$, we get the same relation $x_1 = x_2$. Hence, there is only one linearly independent eigenvector :

$$k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, We solve for $\boldsymbol{\eta}$:

$$\begin{bmatrix}4 & -4\\ 4 & -4\end{bmatrix}\boldsymbol{\eta} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

It has solution in the form

$$\boldsymbol{\eta} = \begin{bmatrix}\frac{1}{4} + \eta_2\\ \eta_2\end{bmatrix}$$

Choose $\eta_2 = 0$, we get

$$\boldsymbol{\eta} = \begin{bmatrix}\frac{1}{4}\\ 0\end{bmatrix}$$

A general solution is, therefore,

$$\mathcal{X} = c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-3t} + c_2\left(\begin{bmatrix}1\\ 1\end{bmatrix}te^{-3t} + \begin{bmatrix}1/4\\ 0\end{bmatrix}e^{-3t}\right)$$

Apply the initial values to find $C_1 = 1$ and $C_2 = -12$. The particular solution is

$$\mathcal{X} = \begin{bmatrix}1\\ 1\end{bmatrix}e^{-3t} - 12\left(\begin{bmatrix}1\\ 1\end{bmatrix}te^{-3t} + \begin{bmatrix}1/4\\ 0\end{bmatrix}e^{-3t}\right) = \begin{bmatrix}-12t-2\\ -12t+1\end{bmatrix}e^{-3t}$$

Transformation of a system of second order linear differential equations to a system of four linear differential equations of the first order

We explain this process through the physical application of the coupled pendula given here after

The coupled pendula

Two identical simple pendula oscillate in the plane as shown in Figure 1. Both pendula consist of light rods of length ℓ = 10 and are suspended from the same ceiling a distance L = 15 apart, with equal masses m = 1 attached to their ends. The angles the pendula make to the downward vertical are $\theta_1$and θ_2, and theyare coupled through the spring shown which has stiffness coefficient k = 1.The spring has unstretched length L = 15.

We assume that the acceleration due to gravity g = 10.

(a) Assuming that the oscillations of the spring remain small in amplitude, so that $|\theta_1| \ll 1$ and $|\theta_2| \ll 1$.
Application of Newton's second law and Hooke's law, shows that the coupled pendula system gives rise to the system of differential equations

$$\theta_1{}'' = -2\theta_1 + \theta_2 \qquad (1)$$

$$\theta_2'' = \theta_1 - 2\theta_2 \qquad (2)$$

It is usual in the literature to put this system of second order differential equations in the matrix – vector form

$$\frac{d^2y}{dt^2} = Ay \qquad (3)$$

Where

$$B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$and \quad y = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Here $B$is known as the stiffness matrix and y is the vector of unknown angles for each of the pendula shown in Figure 7.1.

(b) By looking for a solution of the form

$$\mathrm{y(t)} = \mathrm{c}\, e^{\lambda \mathrm{t}}$$

for a constant vector c, show that solving the system of differential equations (3) reduces to solving the eigenvalue problem

$$(\mathrm{B} - \lambda^2 \mathrm{I})\mathrm{c} = 0 \tag{4}$$

(c) Solve the eigenvalue problem (4) in part (b) above, stating clearly the eigenvalues and associated eigenvectors.

(d) Hence enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue-eigenvector pair.

(e) Finally, write down the general solution of the system of equations (3)

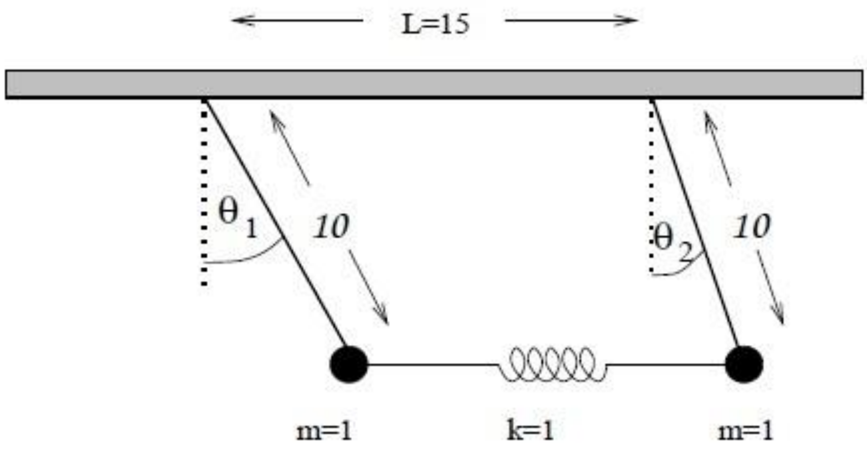

FIGURE 1. Simple coupled pendula system.

Transformation to a linear system of the first order

In the following, we study this problem by transforming the above system of two second order differential equations to an equivalent system of four linear differential equations of the first order as follows

Solution:

We reconsider the system of second order equations :

$$\theta_1{}'' = -2\theta_1 + \theta_2$$
$$\theta_2'' = \theta_1 - 2\theta_2$$

And we define

$$\theta_3 = \theta'_1$$
$$\theta_4 = \theta'_2$$

This leads to ⟹

$$\theta_3^{'} = -2\theta_1 + \theta_2$$

$$\theta_4^{'} = \theta_1 - 2\theta_2$$

Thus, we have produced a system of four equations and four unknowns, $\theta_1, \theta_2,\ \theta_3, \theta_4$

We consider that we have a vector$X$

Where

$$\mathrm{X} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$X^{'} = AX$$

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

To solve this system we find eigen values and eigenvectors of the matrix A. The four eigen values are:

$$\lambda_{1,2} = \pm i \quad , \qquad \lambda_{3,4} = \pm\sqrt{3}i$$

We used the MATLAB software to find the eigenvectors as

$$w_1 = \mathrm{u}_1 + \mathrm{iv}_1 = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5\mathrm{i} \\ -0.5\mathrm{i} \end{bmatrix} and \quad w_2 = u_2 + v_2 = \begin{bmatrix} -.3536i \\ 0.3536i \\ 0.6124 \\ -0.6124 \end{bmatrix}$$

$$p = [\ v_1 \quad u_1 v_2 \quad u_2]$$

$$p = \begin{bmatrix} 0 & -0.5 & -0.3536 & 0 \\ 0 & -0.5 & 0.3536 & 0 \\ -0.5 & 0 & 0 & 0.6124 \\ -0.5 & 0 & 0 & -0.6124 \end{bmatrix}$$

Then

$$p^{-1} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1.4140 & 1.4140 & 0 & 0 \\ 0 & 0 & 0.8165 & -0.8165 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

The solution subject to the initial condition

$$x(0) = x_0$$

is given by

$$x(\mathrm{t}) = P \begin{bmatrix} cost & -sint & 0 & 0 \\ sint & cost & 0 & 0 \\ 0 & 0 & cos\sqrt{3}t & -sin\sqrt{3}\mathrm{t} \\ 0 & 0 & sin\sqrt{3}\mathrm{t} & cos\sqrt{3}t \end{bmatrix} P^{-1}x_0$$

From which, we get the final result as

$$x(\mathrm{t}) =$$

$$\begin{bmatrix} -0.5cost + 0.4cos\sqrt{3}t & -0.5cost - 0.4cos\sqrt{3}t & -0.5sint + 0.4sin\sqrt{3}t & -0.5sint - 0.4sin\sqrt{3}t \\ 0.5cost - 0.4cos\sqrt{3}t & 0.5cost + 0.4cos\sqrt{3}t & 0.5sint + 0.4sin\sqrt{3}t & 0.5sint - 0.4sin\sqrt{3}t \\ 0.5sint - 0.6sin\sqrt{3}t & 0.5sint + 0.6sin\sqrt{3}t & -0.5cost + 0.6cos\sqrt{3}t & -0.5cost - 0.6cos\sqrt{3}t \\ -0.5sint + 0.6sin\sqrt{3}t & -0.5sint - 0.6sin\sqrt{3}t & 0.5cost + 0.6cos\sqrt{3}t & -0.5cost - 0.6cos\sqrt{3}t \end{bmatrix} x_0$$

The vector x_0 represents the initial values of the functions $\theta_1, \theta_2, \theta_3, \theta_4$ and the product of the matrices gives the values of these functions at any instant of time t.
We notice that the third and the forth rows are respectively the derivatives of the first and the second rows and this result indicates the exactness of our solutions since from the beginning we assumed that $\theta_3 = \theta'_1$ and $\theta_4 = \theta'_2$.

The following four functions , appearing as a first row of the square matrix, determine the solution for any initial values of the functions $\theta_1, \theta_2, \theta_3, \theta_4$ between t=0.0 and t=20

$F_1(t) = -0.5cost + 0.4cos\sqrt{3}t$,

$F_2(t) = -0.5cost - 0.4cos\sqrt{3}t$,

$F_3(t) = -0.5sint + 0.4sin\sqrt{3}t$

and

$F_4(t) = -0.5sint - 0.4sin\sqrt{3}t$.

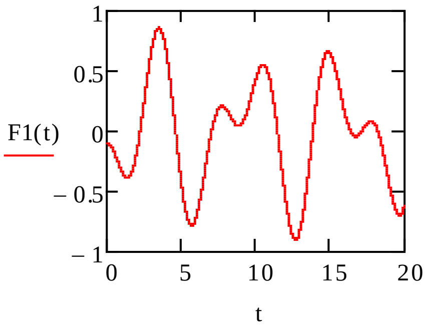

Figure 2: The function F1(t)

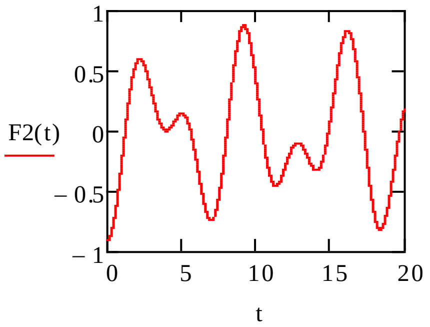

Figure 3: The function F2(t)

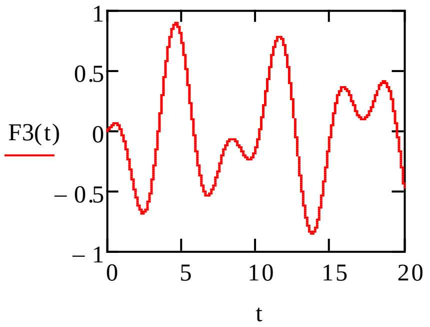

Figure 4: The function F3(t)

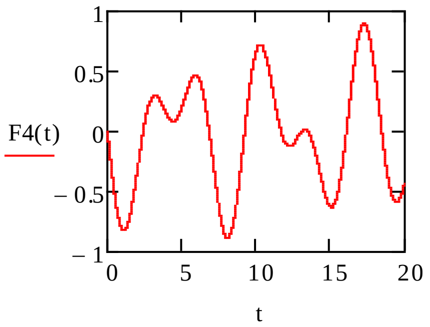

Figure 5: The function F4(t)

Conclusions

The main results of the present work are

(1)presenting how to transform a higher order linear ordinary differential equation to a system of first order linear differential equations.

(2) giving certain applications to make clear the step (1).

(3) Presenting how to replace a system of second order ordinary differential equations by a suitable system of the first order.

(4) Giving a physical application of the coupled pendula as an illustration of the method and presenting two methods

for the solution.

(5) giving the graphs of the functions determining the solution of the coupled pendula for any given initial situations.

Reference

[1] Boyce, W.E. & DiPrima, R.C. 1996 Elementary differential equations and boundary-value problems, John Wiley.

[2] Zill, D.G. 2000 A first course in differential equations: the classic fifth edition, Brooks/Cole Pub Co.

[3] Hildebrand, F.B. 1976 Advanced calculus for applications, Second Edition, Prentice-Hall.